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# TWO-STAGE CUBATURE KALMAN FILTERAND ITS APPLICATION IN WATER POLLUTION MODEL

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## ABSTRACT

Water Pollution Model is a nonlinear system which present the random bias. The most common method is to use augmented state Cubature Kalman Filter, but the computational requirement of augmented state Kalman filter may become excessive. It is easily overflow and fail when running on digital computer. In this paper, two-stage Cubature Kalman filter is proposed to solve this problem. The estimate of two-stage Cubature Kalman filter can be expressed as the output of the advanced bias free filter and bias filter. Contrast augmented state Cubature Kalman filter, two-stage Cubature Kalman filter is equivalent to the augmented state Cubature Kalman Filter in terms of computational accuracy, but computation is much smaller than augmented state Cubature Kalman Filter. The simulation results prove the validity of the two-stage Cubature Kalman filter in Water Pollution Model and prove the equivalence of the two algorithms.

## KEYWORDS

Two-stage Cubature Kalman Filter, Water Pollution Model, Nonlinear system, Random Bias.

## 1. INTRODUCTION

Consider the problem of nonlinear system with randam bias, it is common to use augmented state Cubature Kalman Filter, which treat the bias as part of the state. When the state is estimated, as well as the bias. With the increase of system equation dimension in practical application, the amount of computation for augmented state Cubature Kalman Filter will dramatically increase. It is easily overflow and fail when running on digital computer. To avoid use of augmented state Cubature Kalman Filter, the researchers propose a two-stage filtering method. Friedland proposed a two-stage filter to decouple the filter into two parallel filters-bias free filter and bias filter, but this method is optimal for constant bias, it is suboptimal for random bias unless exist suitable algebraic constraint. A group researcher proposed an optimal two-stage Kalman estimator respect which is an extension of Friedland's estimator and is optimal in general conditions [1-3]. Simplify the computation. Many researchers had also contributed in this area, Chien-Shu Hsieh presented a general two-stage Kalman filter which provides the optimal estimate of the system state and can be applied to general, time-varying and linear dynamic systems [1]. The new filter can reduce the computational burdens.

It is known that in practical applications require nonlinear filter techniques. Chien-Shu Hsieh extended the linear general two-stage filter to nonlinear systems and proposed a general two-stage extended Kalman filter and it is mathematically equivalent to the extend Kalman filter [1]. Other researchers presented a two-stage unscented Kalman filter which is designed by using the forgetting factor to compensate the effects of incomplete information [4]. In other studies, most of researchers proposed a novel two-stage extended Kalman filter algorithm, the proposed approach is respectively applied to estimating bias faults and loss of effectiveness for reaction flywheels in satellite attitude control systems [5]. Some researchers also extended the two-stage method to Cubature Kalman filter and proposed two-stage Cubature Kalman filter, which prevents augmented state Cubature Kalman Filter which brings dimension disaster and solves high-dimensional nonlinear filter problem with minimal computational effort [6].

In the numerical simulation of solute transport in groundwater, there are unavoidable bias: the error of the model itself, the error in the field measurement, and the error in the process of the solution [7]. In order to get closer to the real parameters, identification results and water quality prediction results, it is necessary to minimize the influence of these bias [8].

The Kalman filter algorithm is essentially a minimum variance estimation in the state space. It is applied to the identification of water quality parameters, and the main task is to find the state equations and observation equations [9-11]. The state equation describes the variation of the estimated value (the parameter to be considered) between the current and the next; The observation equation describes the relation between the estimated value and the actual observed value. Through the "prediction correction", the optimal values of parameters are obtained, and the purpose of parameter identification is achieved [12].

## 2. DETERMINATION OF WATER POLLUTION MODEL

### 2.1 Determination of state equation

It is assumed that hydrogeological conditions in the study area remain stable, The horizontal and vertical dispersion coefficients, the seepage velocity and the nitrification / denitrification coefficients remain unchanged [13]. The model parameter of solute transport is regarded as the state vector, and the observed solute concentration is regarded as the observation vector of the system. Then the corresponding state equation is:

$$x_{k+1} = f(x_k) + D_k b_k + \omega_k^x \quad (1a)$$

where$x_{k+1} = (D_x, D_y, u_x, u_y, q_s, R, 1)$ is an unknown parameter vector needed to be identified? the nonlinear function $f(\cdot)$ is state transition function. The noise sequence $\omega_k^x$ is zero mean uncorrelated Gaussian random sequences [14]. The error of the model itself, the error in the field measurement, and the error in the process of the solution are used as the unavoidable bias in the model, the bias equation is:

$$b_{k+1} = b_k + \omega_k^b \quad (1b)$$

where the noise sequence $\omega_k^b$ is zero mean uncorrelated Gaussian random sequences.

### 2.2 Determination of observation equation

Consider the following solute transport equation:

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} - u_x \frac{\partial C}{\partial x} - u_y \frac{\partial C}{\partial y} - q_s C + RC$$

Where:$D_x, D_y$ are dispersion coefficients along the X and Y axis directions, respectively; $u_x, u_y$ are the seepage velocity along the X and Y axis directions, respectively; $q_s$represents source and sink; $R$represents coefficient of decay reaction; $C$represents indicating the concentration of contaminants.

The equation of solute transport in groundwater system is nonlinear, therefore, the Cubature Kalman filter is used to solve the problem. The observation equation is:

$$z_k = h(x_k) + F_k b_k + \upsilon_k \quad (1c)$$

where the nonlinear function $h(\cdot)$ is observation transition function. The noise sequence $\upsilon_k$ is zero mean uncorrelated Gaussian random sequences. The noise sequence $\omega_k^x$, $\omega_k^b$ and $\upsilon_k$ are zero mean uncorrelated Gaussian random sequences with

$$E\left[\begin{bmatrix}\omega_k^x \\ \omega_k^b \\ \upsilon_k\end{bmatrix}\begin{bmatrix}\omega_j^x \\ \omega_j^b \\ \upsilon_j\end{bmatrix}^T\right] = \begin{bmatrix}Q_k^x & 0 & 0 \\ 0 & Q_k^b & 0 \\ 0 & 0 & R_k\end{bmatrix}\delta_{kj} \quad (2)$$

Where $Q_k^x > 0, Q_k^b > 0, R_k > 0$ and $\delta_{kj}$ is the Kronecker delta. The initial states $x_0$and $b_0$ are assumed to be uncorrelated with the white noise processes $\omega_k^x$, $\omega_k^b$and $\upsilon_k$. We assume that the initial conditions $x_0$ and $b_0$ are Gaussian random variables with

$$E[x_0] = \bar{x}_0, E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T] = P_0^x > 0$$
$$E[b_0] = \bar{b}_0, E[(b_0 - \bar{b}_0)(b_0 - \bar{b}_0)^T] = P_0^b > 0$$
$$E\left[(x_0 - \bar{x}_0)(b_0 - \bar{b}_0)^T\right] = P_0^{xb} > 0$$

## 3. AUGMENTED STATE CUBATURE KALMAN FILTER

Define:

$$X_{k+1} = \begin{bmatrix}x_{k+1} \\ b_{k+1}\end{bmatrix}, f(X_k) = \begin{bmatrix}f(x_k) + D_k b_k \\ b_k\end{bmatrix}, \omega_k = \begin{bmatrix}\omega_k^x \\ \omega_k^b\end{bmatrix}$$
$$h(X_k) = h(x_k) + F_k b_k$$

The model given by Eqs. (1a)-(1c) may be rewritten as:

$$X_{k+1} = f(X_k) + \omega_x \quad (3a)$$
$$Z_k = h(X_k) + \upsilon_k \quad (3b)$$

Where

$$W = E(\omega_k \omega_j) = \begin{bmatrix}Q_k^x & 0 \\ 0 & Q_k^b\end{bmatrix}\delta_{kj}$$

According to the Cubature Kalman filter, treating $x_k$ and $b_k$ as the augmented system state, the augmented state Cubature Kalman filter is described by:

*A. Time Update*

1)Assume at time k that the posterior density function $p(x_{k-1|k-1}) = \aleph(\hat{X}_{k-1|k-1}, P_{k-1|k-1})$ is known, factorize

$P_{k-1|k-1} = S_{k-1|k-1} S_{k-1|k-1}^T$ (4)

2)Evaluate the cubature points(i=1,2, ⋯, m)

$X_{i,k-1|k-1} = S_{k-1|k-1}\xi_i + \hat{X}_{k-1|k-1}$ (5)

3)Evaluate the propagated cubature points(i=1,2, ⋯, m)

$X_{i,k-1|k-1}^* = f(X_{i,k-1|k-1}, u_{k-1})$ (6)

4)Estimate the predicted state

$\hat{X}_{k|k-1} = \frac{1}{m}\sum_{i=1}^m X_{i,k-1|k-1}^*$(7)

5)Estimate the predicted error covariance

$P_{k|k-1} = \frac{1}{m}\sum_{i=1}^m X_{i,k-1|k-1}^* {X_{i,k-1|k-1}^*}^T - \hat{X}_{k|k-1}{\hat{X}_{k|k-1}}^T + Q_{k-1}$ (8)

*B. Measurement Update*

Factorize

$P_{k|k-1} = S_{k|k-1} S_{k|k-1}^T$ (9)

1)Evaluate the cubature points(i=1,2, ⋯, m)

$X_{i,k|k-1} = S_{k|k-1}\xi_i + \hat{X}_{k|k-1}$ (10)

2)Evaluate the propagated cubature points(i=1,2, ⋯, m)

$Z_{i,k|k-1} = h(X_{i,k|k-1})$ (11)

3)Estimate the predicted state

$\hat{Z}_{k|k-1} = \frac{1}{m}\sum_{i=1}^m Z_{i,k|k-1}$(12)

4)Estimate the innovation covariance matrix

$P_{zz,k|k-1} = \frac{1}{m}\sum_{i=1}^m Z_{i,k|k-1} {Z_{i,k|k-1}}^T - \hat{Z}_{k|k-1}{\hat{Z}_{k|k-1}}^T + R_k$ (13)

5)Estimate the cross-covariance matrix

$P_{xz,k|k-1} = \frac{1}{m}\sum_{i=1}^m X_{i,k|k-1} {Z_{i,k|k-1}}^T - \hat{X}_{k|k-1}{\hat{Z}_{k|k-1}}^T$ (14)

6)Estimate the Kalman gain

$K_k = P_{xz,k|k-1}{P_{zz,k|k-1}}^{-1}$ (15)

7)Estimate the updated state

$\hat{X}_{k|k} = \hat{X}_{k|k-1} + K_k(Z_k - \hat{Z}_{k|k-1})$ (16)

8)Estimate the corresponding error covariance

$P_{k|k} = P_{k|k-1} - K_k P_{zz,k|k-1} K_k^T$ (17)

The detailed steps of the augmented state Cubature Kalman filter Algorithm are summarized in Table 1.

**Table 1:** The augmented state Cubature Kalman filter Algorithm

| |
|---|
| **Time update** |
| 1) Evaluate $S_{k-1|k-1}$ by factorize $P_{k-1|k-1}$(4). |
| 2) Calculate the cubature points $X_{i,k-1|k-1}$ (5)and the propagated cubature points $X_{i,k-1|k-1}^*$ (6). |
| 3) Estimate the predicted state $\hat{X}_{k|k-1}$ (7). |
| 4) Estimate the predicted error covariance $P_{k|k-1}$ (8). |
| **Measurement update** |
| 1) Factorize $P_{k|k-1}$ gets $S_{k|k-1}$(9). |
| 2)Calculate the cubature points $X_{i,k|k-1}$(10) and the propagated cubature points $Z_{i,k|k-1}$by measurement equation (11). |
| 3) Estimate the predicted measurement $\hat{Z}_{k|k-1}$(12). |
| 4) Estimate the cross-covariance matrix $P_{xz,k|k-1}$(14). |
| 5) Estimate the Kalman gain $K_k$ (15). |
| 6) Estimate the updated state $\hat{X}_{k|k}$(16) and the corresponding error covariance $P_{k|k}$(17). |

The above filter dimension is *n+p*. when p is comparable to n, contrast initial system state dimension, the new state vector $X_k$ dimension becomes substantially larger, the computational requirement of the augmented state Kalman filter may become excessive. It is easily overflow and fail when running on digital computer. To solve this problem, a large number of researchers proposed two-stage filter algorithm, Through the summary of these literatures, in literature, we proposed two-stage Cubature Kalman filter(TSCKF), This method was proved that under an algebraic constraint the two-stage Kalman filter is optimal, but the algebraic constraint is restrictive in practice, so two-stage Cubature Kalman filter is usually suboptimal. It is proposed a two-stage Cubature filterin this paper, is equivalent to the Augmented state Cubature Kalman Filter.

## 4. TWO-STAGE CUBATURE KALMAN FILTER

Theorem 1.Two-stage Cubature Kalman Filter

$\bar{X}_{k|k}^1$is the output of the advancd bias-free filter.

$$\bar{X}_{k|k-1}^1 = \frac{1}{m}\sum_{i=1}^m f^1(S_{k-1|k-1}\xi_i + T(\Psi, \bar{X}_{k-1|k-1}), u_{k-1}) - \Phi(\bar{X}_{k|k-1}^2)$$
$$\bar{X}_{k|k}^1 = \bar{X}_{k|k-1}^1 + \Phi(\bar{X}_{k|k-1}^2) + V_k(\bar{X}_{k|k}^2 - \bar{X}_{k|k-1}^2) - \Psi(\bar{X}_{k|k}^2)$$
$$+ \bar{K}_k^1\left(Z_k - \frac{1}{m}\sum_{i=1}^m h(S_{k|k-1}\xi_i + T(\Phi, \bar{X}_{k|k-1}), u_k)\right)$$
$$\bar{P}_{k|k-1}^1 = M_{k-1}^{11} + Q_{k-1}^{11} - U_k(M_{k-1}^{12} + Q_{k-1}^{12})^T$$
$$\bar{P}_{k|k}^1 = \bar{P}_{k|k-1}^1 + U_k\bar{P}_{k|k-1}^2 U_k^T - V_k\bar{P}_{k|k-1}^2 V_k^T - \bar{K}_k^1 P_{zz,k|k-1}(\bar{K}_k^1)^T$$
$$- \bar{K}_k^1 P_{zz,k|k-1}(\bar{K}_k^2)^T V_k^T - (\bar{K}_k^1 P_{zz,k|k-1}(\bar{K}_k^2)^T V_k^T)^T$$
$$\bar{K}_k^1 = N_k^1 - V_k N_k^2$$

$\bar{X}_{k|k}^2$is the output of the bias filter.

$$\bar{X}_{k|k-1}^2 = \frac{1}{m}\sum_{i=1}^m f^2(S_{k-1|k-1}\xi_i + T(\Psi, \bar{X}_{k-1|k-1}), u_{k-1})$$
$$\bar{X}_{k|k}^2 = \bar{X}_{k|k-1}^2 + \bar{K}_k^2\left(Z_k - \frac{1}{m}\sum_{i=1}^m h(S_{k|k-1}\xi_i + T(\Phi, \bar{X}_{k|k-1}), u_k)\right)$$

$$\bar{P}^2_{k|k-1} = M^{22}_{k-1} + Q^{22}_{k-1}$$
$$\bar{P}^2_{k|k} = \bar{P}^2_{k|k-1} - \bar{K}^2_k P_{zz,k|k-1}(\bar{K}^2_k)^T$$
$$\bar{K}^2_k = N^2_k$$

The blending matrices $U_k$ and $V_k$ are given as follows:

$$U_k = (M^{12}_{k-1} + Q^{12}_{k-1})(M^{22}_{k-1} + Q^{22}_{k-1})^{-1}$$
$$V_k = U_k - \bar{K}^1_k P_{zz,k|k-1}(\bar{K}^2_k)^T(\bar{P}^2_{k|k-1})^{-1}$$

Proof. The key idea for advanced Two-stage Cubature Kalman Filter is based on state transformations that make the covariance matrices block diagonal.

In the linear systems, two-stage Kalman Filter can be obtained by the following $T$transformation:

$$T(G) = \begin{pmatrix} I_{n-p} & G \\ 0 & I_p \end{pmatrix} (18)$$

Thus, using the two-stage transformations, Cubature Kalman Filter can become the following form:

$$\hat{X}_{k|k-1} = T(U_k)\bar{X}_{k|k-1} (19)$$
$$\hat{X}_{k|k} = T(V_k)\bar{X}_{k|k} \quad (20)$$
$$P_{k|k-1} = T(U_k)\bar{P}_{k|k-1}T^T(U_k) (21)$$
$$P_{k|k} = T(V_k)\bar{P}_{k|k}T^T(V_k) \quad (22)$$
$$K_k = T(V_k)\bar{K}_k \quad (23)$$

where $\bar{P} = diag\{\bar{P}^1, \bar{P}^2\}$.

To extend the two-stage transformations to nonlinear system, the $T$ transformation of (18) is proposed as following:

$$T(F, X) = \begin{bmatrix} X_1 + F(X_2) \\ X_2 \end{bmatrix} \quad (24)$$

where $X = \{(X^1)^T, (X^2)^T\}^T$ in which $X^1 \in R^{n-p}$ and $X^2 \in R^p$, and $F(X^2)$ is a nonlinear function of the substate $X^2$.

From (24), it have the following properties:

$$\frac{\partial T(\Phi, \bar{X}_{k|k-1})}{\partial \bar{X}_{k|k-1}} = \begin{bmatrix} I_{n-p} & U_k \\ 0 & I_p \end{bmatrix} \equiv T(U_k) (25)$$
$$\frac{\partial T(\Psi, \bar{X}_{k|k})}{\partial \bar{X}_{k|k}} = \begin{bmatrix} I_{n-p} & V_k \\ 0 & I_p \end{bmatrix} \equiv T(V_k) \quad (26)$$

Where

$$U_k = \frac{\partial \Phi(\bar{X}^2_{k|k-1})}{\partial \bar{X}^2_{k|k-1}}, V_k = \frac{\partial \Psi(\bar{X}^2_{k|k})}{\partial \bar{X}^2_{k|k}} \quad (27)$$

Using the T transformation with (24), the two-stage transformation(19)-(23) then become

$$\hat{X}_{k|k-1} = T(\Phi, \bar{X}_{k|k-1}) (28)$$
$$\hat{X}_{k|k} = T(\Psi, \bar{X}_{k|k}) \quad (29)$$
$$P_{k|k-1} = \frac{\partial T(\Phi, \bar{X}_{k|k-1})}{\partial \bar{X}_{k|k-1}} \bar{P}_{k|k-1} \left(\frac{\partial T(\Phi, \bar{X}_{k|k-1})}{\partial \bar{X}_{k|k-1}}\right)^T \quad (30)$$
$$P_{k|k} = \frac{\partial T(\Psi, \bar{X}_{k|k})}{\partial \bar{X}_{k|k}} \bar{P}_{k|k-1} \left(\frac{\partial T(\Psi, \bar{X}_{k|k})}{\partial \bar{X}_{k|k}}\right)^T \quad (31)$$
$$K_k = \frac{\partial T(\Psi, \bar{X}_{k|k})}{\partial \bar{X}_{k|k}} \bar{K}_k \quad (32)$$

where $\Phi$ and $\Psi$ are two determined nonlinear functions.

Next, based on the above (28)-(32), the two-stage Cubature Kalman filter can be obtained via the following method.

At the first step, substituting (7),(16) into the left-hand side of (28),(29) and using(24), We obtain

$$\begin{bmatrix} \bar{X}^1_{k|k-1} + \Phi(\bar{X}^2_{k|k-1}) \\ \bar{X}^2_{k|k-1} \end{bmatrix} = \frac{1}{m}\sum_{i=1}^m f\left(S_{k-1|k-1}\xi_i + T(\Psi, \bar{X}_{k-1|k-1}), u_{k-1}\right) (33)$$

$$\begin{bmatrix} \bar{X}^1_{k|k} + \Psi(\bar{X}^2_{k|k}) \\ \bar{X}^2_{k|k} \end{bmatrix} = \begin{bmatrix} \bar{X}^1_{k|k-1} + \Phi(\bar{X}^2_{k|k-1}) \\ \bar{X}^2_{k|k-1} \end{bmatrix} + K_k\left(Z_k - \frac{1}{m}\sum_{i=1}^m h\left(S_{k|k-1}\xi_i + T(\Phi, \bar{X}_{k|k-1}), u_k\right)\right) (34)$$

Expanding(33),(34)and using (26),(32)gets

$$\bar{X}^1_{k|k-1} = \frac{1}{m}\sum_{i=1}^m f^1\left(S_{k-1|k-1}\xi_i + T(\Psi, \bar{X}_{k-1|k-1}), u_{k-1}\right) - \Phi(\bar{X}^2_{k|k-1}) (35)$$

$$\bar{X}^1_{k|k} = \bar{X}^1_{k|k-1} + \Phi(\bar{X}^2_{k|k-1}) + V_k(\bar{X}^2_{k|k} - \bar{X}^2_{k|k-1}) - \Psi(\bar{X}^2_{k|k}) + \bar{K}^1_k\left(Z_k - \frac{1}{m}\sum_{i=1}^m h\left(S_{k|k-1}\xi_i + T(\Phi, \bar{X}_{k|k-1}), u_k\right)\right) (36)$$

$$\bar{X}^2_{k|k-1} = \frac{1}{m}\sum_{i=1}^m f^2\left(S_{k-1|k-1}\xi_i + T(\Psi, \bar{X}_{k-1|k-1}), u_{k-1}\right) (37)$$

$$\bar{X}^2_{k|k} = \bar{X}^2_{k|k-1} + \bar{K}^2_k\left(Z_k - \frac{1}{m}\sum_{i=1}^m h\left(S_{k|k-1}\xi_i + T(\Phi, \bar{X}_{k|k-1}), u_k\right)\right) (38)$$

where

$$f(\cdot) = [(f^1(\cdot))^T \quad (f^2(\cdot))^T]^T$$
$$\bar{K}_k = [(\bar{K}^1_k)^T \quad (\bar{K}^2_k)^T]^T$$

According to (8), order

$$M_{k-1} = \frac{1}{m}\sum_{i=1}^m X^*_{i,k-1|k-1} \sum_{i=1}^m X^*_{i,k-1|k-1}{}^T - \hat{X}_{k|k-1}\hat{X}_{k|k-1}{}^T \quad (39)$$

there is

$$M_{k-1} = \begin{bmatrix} M^{11}_{k-1} & M^{12}_{k-1} \\ (M^{12}_{k-1})^T & M^{22}_{k-1} \end{bmatrix} (40)$$

obtain

$$P_{k|k-1} = M_{k-1} + Q_{k-1} = \begin{bmatrix} M^{11}_{k-1} + Q^{11}_{k-1} & M^{12}_{k-1} + Q^{12}_{k-1} \\ (M^{12}_{k-1} + Q^{12}_{k-1})^T & M^{22}_{k-1} + Q^{22}_{k-1} \end{bmatrix} (41)$$

using (25)(30) yields

$$\bar{P}^1_{k|k-1} = M^{11}_{k-1} + Q^{11}_{k-1} - U_k(M^{12}_{k-1} + Q^{12}_{k-1})^T \quad (42)$$
$$\bar{P}^2_{k|k-1} = M^{22}_{k-1} + Q^{22}_{k-1} (43)$$
$$U_k = (M^{12}_{k-1} + Q^{12}_{k-1})(M^{22}_{k-1} + Q^{22}_{k-1})^{-1} (44)$$

Transformation formula(31)and expanding by(30)(32)

$$\bar{P}^1_{k|k} = \bar{P}^1_{k|k-1} + U_k\bar{P}^2_{k|k-1}U_k^T - V_k\bar{P}^2_{k|k-1}V_k^T - \bar{K}^1_k P_{zz,k|k-1}(\bar{K}^1_k)^T - \bar{K}^1_k P_{zz,k|k-1}(\bar{K}^2_k)^T V_k^T - \left(\bar{K}^1_k P_{zz,k|k-1}(\bar{K}^2_k)^T V_k^T\right)^T (45)$$
$$\bar{P}^2_{k|k} = \bar{P}^2_{k|k-1} - \bar{K}^2_k P_{zz,k|k-1}(\bar{K}^2_k)^T (46)$$
$$V_k = U_k - \bar{K}^1_k P_{zz,k|k-1}(\bar{K}^2_k)^T(\bar{P}^2_{k|k-1})^{-1} (47)$$

According to (13)-(15), order

$$N_k = P_{xz,k|k-1}P^{-1}_{zz,k|k-1} (48)$$

we have

$$K_k = N_k = \begin{bmatrix} N^1_k \\ N^2_k \end{bmatrix} (49)$$

using(29)

$$\bar{K}^1_k = N^1_k - V_k N^2_k (50)$$
$$\bar{K}^2_k = N^2_k (51)$$

are deduced.

The proof is finished. It remains to solve the problem of obtaining $\Phi$ and $\Psi$. This can be done by using (32) and the backward difference equation as follows:

$$\Phi(\bar{X}^2_{k|k-1}) = \Phi(\bar{X}^2_{k-1|k-2}) + U_k(\bar{X}^2_{k|k-1} - \bar{X}^2_{k-1|k-2})$$
$$\Psi(\bar{X}^2_{k|k}) = \Psi(\bar{X}^2_{k-1|k-1}) + V_k(\bar{X}^2_{k|k} - \bar{X}^2_{k-1|k-1})$$

The Two-stage Cubature Kalman Filter Algorithm are summarized in Table 2.

**Table2:** The Two-stage Cubature Kalman Filter Algorithm

| |
|---|
| **Time update** |
| 1) Evaluate $S_{k-1|k-1}$ by factorize $P_{k-1|k-1}$(4). |
| 2) Calculate the cubature points $X_{i,k-1|k-1}$ (5)and the propagated cubature points$X^*_{i,k-1|k-1}$(6). |
| 3) Estimate the predicted state $\bar{X}^1_{k|k-1}$ (35) and $\bar{X}^2_{k|k-1}$ (37). |
| 4) Estimate the predicted error covariance $P^1_{k|k-1}$ (42) , $P^2_{k|k-1}$ (43) use $M_{k-1}$(40) and $U_k$ in (45). |
| **Measurement update** |
| 1) Factorize $P_{k|k-1}$ gets $S_{k|k-1}$. |
| 2)Calculate the cubature points $X_{i,k|k-1}$(10) and the propagated cubature points $Z_{i,k|k-1}$by measurement equation (11). |
| 3) Estimate the predicted measurement $\hat{Z}_{k|k-1}$(12). |
| 4) Estimate the innovation covariance matrix $P_{zz,k|k-1}$(13) and the cross-covariance matrix$P_{xz,k|k-1}$(14). |
| 5) Estimate the Kalman gain $\bar{K}^1_k$(50) and $\bar{K}^2_k$(51) with $N_k$(48). |
| 6) Estimate the updated state $\bar{X}^1_{k|k}$(36) and $\bar{X}^2_{k|k}$(38). |
| 7)Estimate the corresponding error covariance $P^1_{k|k}$(45), $P^2_{k|k}$(46) and $V_k$ in (47). |

## 5. SIMULATION EXAMPLES

The true value and estimate value of the dispersion coefficients along the X and Y axis directions, the seepage velocity along the X and Y axis directions, and the source and sinkt are given in Figure 1 to Figure 10, as well as estimation error. It can be seen from Figure 2,4,6,8,10 that the estimated error value is within a small range which is too small to be neglected for practical applications. In fact, this error is due to the numerical computer error. Therefore, it is concluded that the estimation accuracy of Two-stage Cubature Kalman Filter and state value is the same, the estimation results can be accepted.

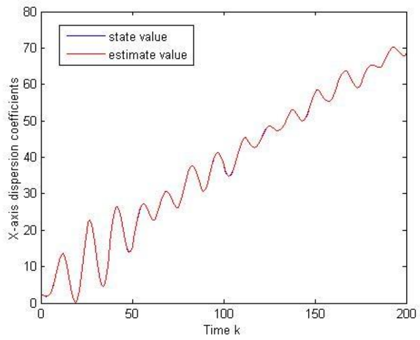



**Figure 1:** X-axis dispersion coefficients

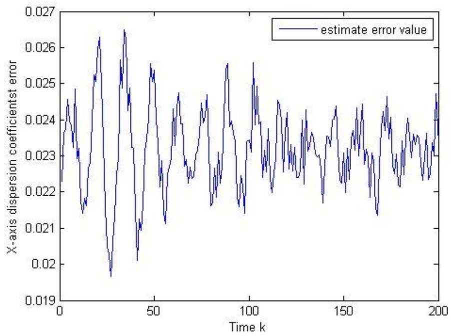



**Figure 2:** X-axisdispersion coefficients error

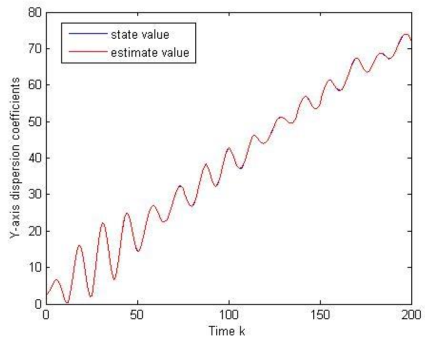



**Figure 3:** Y-axis dispersion coefficients

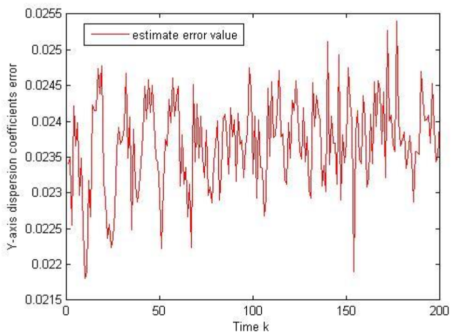



**Figure 4:** Y-axis dispersion coefficientserror

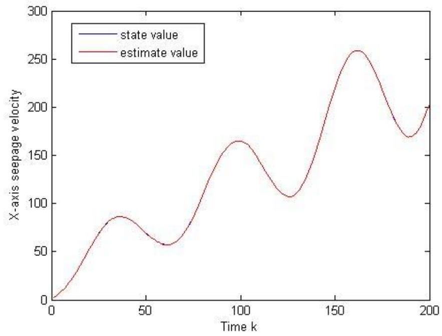



**Figure 5:** X-axis seepage velocity

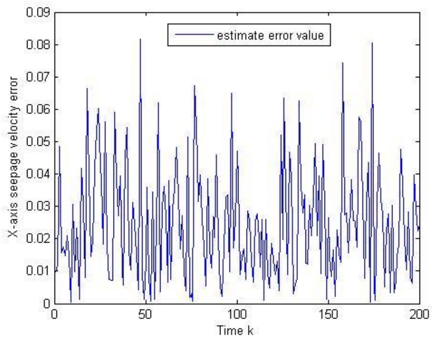
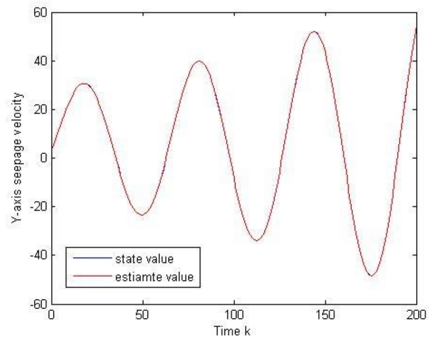



**Figure 6:** X-axis seepage velocityerror




**Figuer 7:** Y-axis seepage velocity

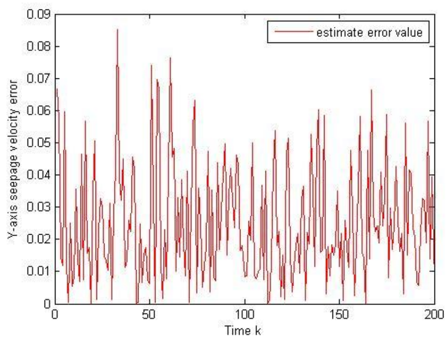



**Figure 8:** Y-axis seepage velocityerror

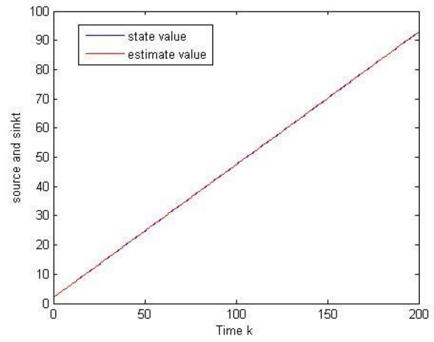



**Figure 9:** Source and sinkt

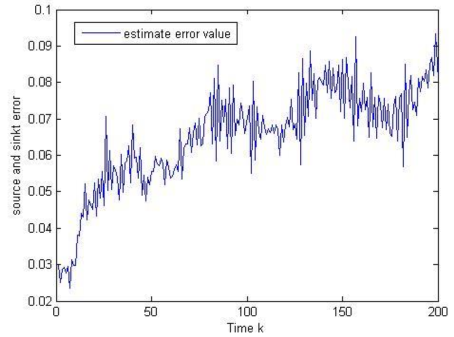



**Figure 10:** Source and sinkt error

Figure 11 and Figure 12 show the bias value and the bias error value. The state values are derived from the augmented state computation and the estimate values are derived from Two-stage Cubature Kalman Filter. The same as the above estimate error figures, the bias estimate error is within a small range and it can be obtained that the estimate value of the bias is similar as the bias state value.

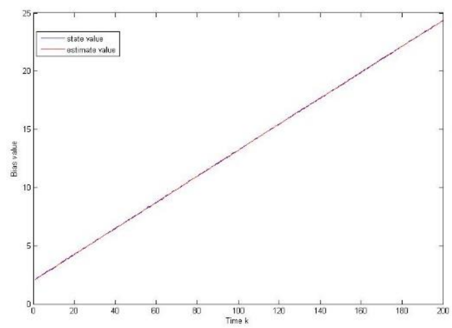



**Figure 11:** Bias value

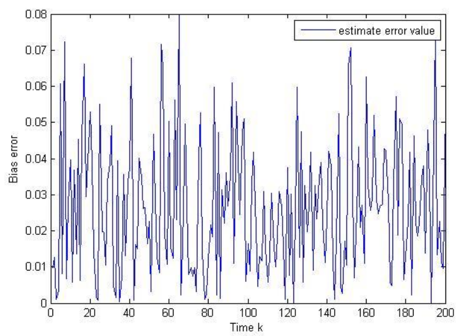



**Figure 12:** Bias error

## 6. CONCLUSION

In this paper, two-stage Cubature Kalman filter is proposed in water pollution model which to solve the nonlinear system with random bias. Contrast previous augmented stateCubature Kalman filter, two-state Cubature Kalman filter is equivalent to the augmented state Cubature Kalman Filter and is optimal. The simulation results prove the validity of the two-stateCubature Kalman filter algorithm and prove the equivalence of the two algorithms.

## ACKNOWLEDGMENTS

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