Fahad Jamal1, Arif Mehmood Khattak2, Gulzar Ali Khan3, Saleem Abdullah2

1Department of Mathematics and Statistics, Riphah International University Islamabad, Pakistan.
2Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan.
3Department of Mathematics, Qurtuba University Peshawar, Pakistan.

*Corresponding Author Email: mehdaniyal@gmail.com

ABSTRACT

In 1999, Russian mathematician Molodtsov planted the novel notion of a soft set which can be supposed as a new mathematical approach for imprecise. Topological structures of soft set have been studied by some mathematicians in current years. M. Shabir and M. Naz presented soft topological spaces and studied some key properties of them. Soft Quad topological spaces are nothing but just Extension of soft topological spaces. When with soft single topology three more soft topologies are huggd together through mathematical discipline then the notion of soft Quad topology will automatically come in play. The central objective of this article is to publicize soft separations axioms in soft quad topological spaces. Moreover to discuss the linkage among them through two paths.

KEYWORDS

Soft quad topological spaces soft $qT_0$ space, soft $qT_1$ space, soft $qT_2$ space, soft $qT_3$ space and soft $qT_4$ space.

1. INTRODUCTION

Many practical problems in economics, engineering, environmental sciences, social sciences, medical sciences etc. cannot be dealt by classical methods because classical methods have inherent difficulties. The reason for these difficulties may be due to the insufficiency of the theories of parameterization tools. Numerous theories exist, for example, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory, i.e., which can be considered as a mathematical tool for dealing with uncertainties [1-3].

Each of these theories has its inherent difficulties as what were pointed out by Molodtsov [4]. Molodtsov originated absolutely a new methodology for modelling uncertainties and applied successfully in directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, and so on [4]. A group researcher gave the first practical application of soft sets in decision making problems [5]. The algebraic structure of set theories dealing with uncertainty is an important problem. Many researchers have contributed towards the algebraic structure of soft set theory. Some of researchers defined soft groups and deduced their fundamental characterizations [6]. Others group of researchers introduced initial concepts of soft rings [7]. Besides, in other studies, researchers defined soft semi rings and several related notions to establish a connection between soft sets and semi rings [8]. Some researchers defined soft modules and investigated their basic properties [9]. A studied soft ideal over a semi groups which characterized generalized fuzzy ideals and fuzzy ideals with thresholds of a semi groups [10]. There are two researchers introduced fuzzy soft modules and intuitionistic fuzzy soft modules and investigated some basic properties [11,12].

Shabir and Naz firstly introduced the notion of soft topological space which are defined over an initial universe with a fixed set of parameters and showed that a soft topological space gives a parameterized family of topological spaces [13]. Theoretical studies of soft topological spaces have also been by some authors in [14-17]. In these studies the concept of soft point is given almost same. Consequently, some results of classical topology are not valid in soft topological spaces. A group researcher introduced the concepts of supra topological spaces, supra closed sets, supra open sets and supra continuous mapping [18]. Many results of topological spaces remain valid in supra topological spaces, but some become false. As a generalized of soft topological spaces, the notion of supra soft topological spaces by dropping only the soft intersection condition [19]. They extend some types of different kinds of subsets of soft topological spaces using the notion of $\gamma$-operation to supra soft topological spaces and study the decompositions of some forms of supra soft continuity. The notion of soft bi topological space which is defined over an initial universal set $X$ with fixed set of parameters and studied some types of soft separation axioms [20]. It is clear that a soft bi topological space is a generalization of a soft topological space as every soft bi topological space is a soft topology while a soft topology is a special case of a soft bi topological space [11].

In the present study, we first give some basic ideas about soft sets and the results already studied. In addition to these, we introduce the concept of soft point to the theory of soft set. According to this definition, we investigate some important notions of soft topological spaces. Later we give separability axioms in soft Quad topological spaces and study some of their characterizations. The separability axioms are the soft separation axioms in Soft Quad topological spaces with respect to ordinary and soft points.

We expect that these results will be of greater importance for the forthcoming learning on Soft Quad topological spaces to accomplish general framework for the most practical applications and to answer the most intricate problems containing doubt in economics, engineering, medical environment and in general.

2. PRELIMINARIES

In this section we will introduce necessary definitions and theorems for soft sets. Molodtsov defined the soft set in the following way. Let $X$ be an initial universe and $E$ be a set of parameters. Let $P(X)$ denotes the power set of $X$ [14].

Definition 2.1. ([4]) A pair $(F, E)$ is called a soft set over $X$; where $F$ is a mapping given by $F : E \rightarrow P(X)$.

In other words, the soft set is a parameterized family of subsets of the set $X$ for $e \in E$; $F(e)$ may be considered as the set of $e$-elements of the soft
set(\(F, E\)); or as the set of \(e - \) approximate elements of the soft set, i.e. 
\((F, E) = \{(e, F(e)) : e \in E, F : E \rightarrow P(X)\}.
\)
After this, \(SS(X)_E\) denotes the family of all soft sets over \(X\) with a fixed set of parameter \(E\).

**Definition 2.2. ([21])** For two soft sets \((F, E)\) and \((G, E)\) over \(X, (F, E)\) is called a sub soft set of \((G, E)\) if \(\forall e \in E, F(e) \subseteq G(e)\). This relationship is denoted by \((F, E) \subseteq (G, E)\).
Similarly, \((F, E)\) is called a soft superset of \((G, E)\) if \((G, E)\) is a soft subset of \((F, E)\). This linkage is denoted by \((F, E) \supseteq (G, E)\). Two soft sets \((F, E)\) and \((G, E)\) over \(X\) are called soft equal if \((F, E)\) is a soft subset of \((G, E)\) and \((G, E)\) is a soft subset of \((F, E)\).

**Definition 2.3. ([21])** The intersection of two soft sets \((F, E)\) and \((G, E)\) over \(X\) is the soft set 
\((\mathcal{H}, E), \text{ where } \forall e \in E, \quad H(e) = F(e) \cap G(e)\). This is denoted by \((F, E) \cap (G, E) = (\mathcal{H}, E)\).

**Definition 2.4. ([21])** The union of two soft sets \((F, E)\) and \((G, E)\) over \(X\) is the soft set \((\mathcal{H}, E), \text{ where } \forall e \in E, \quad H(e) = F(e) \cup G(e)\). This is denoted by \((F, E) \cup (G, E) = (\mathcal{H}, E)\).

**Definition 2.5. ([22])** A soft set \((F, E)\) over \(X\) is said to be a null soft set denoted by \(\varnothing\) if for all \(e \in E\), \(F(e) = \varnothing\).

**Definition 2.6. ([22])** A soft set \((F, E)\) over \(X\) is said to be an absolute soft set denoted by \(\overline{F}\) if for all \(e \in E\), \(F(e) = X\).

**Definition 2.7. ([23])** The soft set \((F, A) \in SS(X)_E\) is called a point soft set in \(X_E\), denoted by \(e_F\), if for the element \(e \in A, F(e) \neq \varnothing\) and \(F(e') = \varnothing\) if for all \(e' \in E - \{e\}\).

**Definition 2.8. ([22])** The complement of a soft set \((F, E)\), denoted by \((\overline{F}, E')\), is defined as \((\overline{F}, E') = (F^c, E)\), where \(F^c : E \rightarrow P(X)\) is a mapping given by \(F^c(e) = X \setminus F(e)\). \(\forall e \in E\) and \(F^c\) is called the soft complement function of \(F\).

**Definition 2.9. ([22])** Let \(Y\) be a non-empty subset of \(X\), then \(\tilde{Y}\) denotes the soft set \((Y, E)\) over \(X\) for which \((Y, E)\) is \(E\) for all \(e \in E\).

**Definition 2.10. ([22])** Let \((F, E)\) be a soft set over \(X\) and \(Y\) be a non-empty subset of \(X\). Then the sub soft set of \((F, E)\) over \(Y\) denoted by \((F, E)'\), is defined as follows:
\(F'(e) = Y \cap F(e)\), \(\forall e \in E\).
In other words \((F, E)' = \tilde{Y} \cap (F, E)\).

**Definition 2.11. ([22])** Let \(\mathcal{S}\) be the collection of soft sets over \(X\) then \(\mathcal{S}\) is said to be a soft Topology on \(X\) if 
\(\mathcal{S}\) is closed under \(\cap\) and \(\cup\).
\(\mathcal{S}\) is a union of any number of soft sets in \(\mathcal{S}\) belongs to \(\mathcal{S}\).
The triplet \((X, \mathcal{S}, \mathcal{E})\) is called a soft topological space over \(X\).

**Proposition 2.12. ([22])** Let \((X, \mathcal{S}, \mathcal{E})\) be a soft topological space over \(X\). Then the collection 
\(\mathcal{S}_E = \{F(e) : (F, E) \in \mathcal{S}\}\) for each \(e \in E\), defines a topology on \(X\).

**Definition 2.13. ([22])** Let \((X, \mathcal{S}, \mathcal{E})\) be a soft topological space over \(X\) and \((F, E)\) be a soft set over \(X\). Then the soft closure of \((F, E)\), denoted by \((\overline{F}, E)\), is the intersection of all soft closed super sets of \((F, E)\). Clearly \((\overline{F}, E)\) is the smallest soft closed set over \(X\) which contains \((F, E)\).

**Proposition 1. ([22])** Let \((X, E, F)\) be a soft topological space over \(X\). If \((X, E, F)\) is soft \(T_2\) space, then for all \(x \in X, x_E = (x, E)\) is closed soft set.

**Proposition 2. ([22])** Let \((Y, \tau, E)\) be a soft sub space of a soft topological space \((X, \tau, E)\) and \((F, S)\) is SS then
1. If \((F, E)\) is soft open set in \(Y\) and \(Y \in \tau,\) then \((F, E) \in \tau\).
2. \((F, E)\) is soft open set in \(Y\) if and only if \((F, E) = Y \cap (G, E)\) for some \((G, E) \in \tau\).
3. \((F, E)\) is soft closed set in \(Y\) if and only if \((F, E) = Y \cap (H, E)\) for some \((H, E) \in \tau\) soft closed set.
hand q-open) if \((F, E) \subseteq \tau_1 \cup \tau_2 \cup \tau_3 \cup \tau_4\) and its complement is said to be soft q-closed.

**Definition 4.2.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a quad soft topological space over \(X, (G, E)\) be a soft set over \(X\) and \(x_0 \in X\). Then \(x\) is said to be a soft \(q\) interior point of \((G, E)\) if there exists soft \(q\)-open set \((F, E)\) such that \(x_0 \in (F, E) \subseteq (G, E)\).

**Definition 4.3.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a soft quad topological space over \(X, (G, E)\) be a soft set over \(X, x_0 \in X\). Then \((G, E)\) is said to be a soft \(q\) neighborhood of \(x\) if there exists soft \(q\)-open set \((F, E)\) such that \(x_0 \in (F, E) \subseteq (G, E)\).

**Proposition 7.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a soft quad topological space over \(X\) and each \(x_n \in X\) has a soft \(q\) neighborhood.

If \((F, E)\) and \((G, E)\) are soft \(q\) neighborhoods of some \(x_n \in X\), then \((F, E) \subseteq (G, E)\) is also a soft \(q\) neighborhood of \(x_n\).

**Proof.** Consider \(x \in (F, E) \subseteq (G, E)\). Then \(x \in (G, E)\), and hence \(x \in (F, E) \subseteq (G, E)\), so \((F, E)\) is a soft \(q\) neighborhood of \(x\).

**Definition 5.5.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a soft quad topological space over \(X\) and \(x_n \in X\) such that \(x \in (F, E) \subseteq (G, E)\). If we can find two soft \(q\)-open sets \((F, E)\) and \((G, E)\) such that \(x \in (F, E) \subseteq (G, E)\) and \((F, E) \cap (G, E) = \emptyset\), then \((X, \tau, t, t_0, t_3, t_4)\) is a called soft \(q\)-closed set. A soft quad \(q\)-closed set is also a soft \(q\)-regular set.

**Definition 5.6.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a soft quad topological space over \(X\) and \((F, E)\) be a soft \(q\)-open set in \(X\) such that \(x \in (F, E) \subseteq (G, E)\). If there exists soft \(q\)-open sets \((F, E)\) and \((G, E)\) such that \(x \in (F, E) \subseteq (G, E)\) and \((F, E) \cap (G, E) = \emptyset\), then \((X, \tau, t, t_0, t_3, t_4)\) is called soft \(q\)-closed set. A soft \(q\)-closed space is also a soft \(q\)-regular space.

**Proof.** Let \((X, \tau, t, t_0, t_3, t_4)\) be a soft quad topological space over \(X\). Then \((X, \tau, t, t_0, t_3, t_4)\) is soft \(q\)-closed set if and only if for each soft \(q\)-open set \((F, E)\) such that \(x \in (F, E) \subseteq (G, E)\) and \((F, E) \cap (G, E) = \emptyset\), then \((X, \tau, t, t_0, t_3, t_4)\) is soft \(q\)-closed set.
Proosition 11. Let \((X, t, t_1, t_2, t_3)\) be a soft quad topological space over \(X\). Then, if \((X, t, t_1, t_2, t_3)\) is soft \(T_3\) space then \((X, t_1, t_2, t_3)\) is pair wise soft \(T_3\) space.

Proof: Suppose \((X, t_1, t_2, t_3)\) is soft space with respect to \(t_1, t_2, t_3\). Let \(a \in X\) be any point then there exists \(t_1, t_2, t_3\) such that \(a \in t_1 \cap t_2 \cap t_3\). Hence \((X, t_1, t_2, t_3)\) is pair wise soft \(T_3\) space.
Since $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is pair wise soft β normal space. So there exists soft open sets $(H_1, E)$ and $(H_2, E)$ such that $(H_1, E)$ is soft open set in $t_1 \cup t_2$ and $(H_2, E)$ is soft open set in $t_3 \cup t_4$ such that

$$(G_6, E) \subseteq (H_1, E)$$

$$(G_6, E) \subseteq (H_2, E)$$

$$(H_1 \cap E) \subseteq H_2 \subseteq E$$

Since $(G_6, E) \subseteq (H_1, E)$ and $(G_6, E) \subseteq (H_2, E)$ are soft open sets in $Y$ there fore $t_1 \cup t_2$ is soft normal space with respect to $t_1 \cup t_2$. Similarly, let $(G_1, E), (G_2, E)$ be soft closed sub sets in $Y$ such that

$$(G_1, E) \subseteq (G_6, E) \subseteq (H_1, E)$$

And

$$(G_2, E) \subseteq (G_6, E) \subseteq (H_2, E)$$

From Proposition 2. Since, $Y$ is soft closed sub set in $X$ then $(G_1, E), (G_2, E)$ are soft closed sets in $X$ such that

$$(G_1, E) \subseteq (G_6, E) \subseteq (H_1, E)$$

Since $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is pair wise soft β normal space so there exists soft open sets $(H_1, E)$ and $(H_2, E)$ such that

$$(H_1, E) \subseteq (H_2, E)$$

Since $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is pair wise soft β normal space in $Y$ there fore $t_1 \cup t_2$ is soft normal space with respect to $t_1 \cup t_2$

$$(Y, t_1, t_2, t_3, t_4, t_5, t_6)$$

6. SOFT AXIOMS WITH RESPECT TO SOFT POINTS

In this section, we brought out soft topological structures known as separation axioms in quad soft topology with respect to soft points. With the applications of these soft separation axioms different result are discussed

**Definition 7.1:** In a soft quad topological space $(X, t_1, t_2, t_3, t_4, t_5, t_6)$

1) $t_1 \cup t_2$ is said to be soft $T_0$ space with respect to $t_1 \cup t_2$ if for each pair of distinct points $e \in E$ and $e' \in E$ there exists $t_1 \cup t_2$ soft open set $(F, E)$ and a $t_2 \cup t_3$ soft open set $(G, E)$ such that $e \in (F, E)$ and $e' \in (G, E)$. Similarly, $t_2 \cup t_3$ is said to be soft $T_0$ space with respect to $t_2 \cup t_3$ if for each pair of distinct points $e \in E$ and $e' \in E$ there exists $t_2 \cup t_3$ soft open set $(F, E)$ and a $t_3 \cup t_4$ soft open set $(G, E)$ such that $e \in (F, E)$ and $e' \in (G, E)$ and $e' \in E$ and $e \in (G, E)$. Soft quad topological space $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is said to be pair wise soft $T_0$ space if $t_1 \cup t_2$ is soft $T_0$ space with respect to $t_1 \cup t_2$ and $t_2 \cup t_3$ is soft $T_0$ space with respect to $t_2 \cup t_3$.

2) $t_1 \cup t_2$ is said to be soft $T_1$ space with respect to $t_1 \cup t_2$ if for each pair of distinct points $e \in E$ and $e' \in E$ there exists $t_1 \cup t_2$ soft open set $(F, E)$ and a $t_3 \cup t_4$ soft open set $(G, E)$ such that $e \in (F, E)$ and $e' \in (G, E)$ and $e' \in E$ and $e \in (G, E)$. Soft quad topological space $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is said to be pair wise soft $T_1$ space if $t_1 \cup t_2$ is soft $T_1$ space with respect to $t_1 \cup t_2$ and $t_2 \cup t_3$ is soft $T_1$ space with respect to $t_2 \cup t_3$.

3) $t_1 \cup t_2$ is said to be soft $T_2$ space with respect to $t_1 \cup t_2$ if for each pair of distinct points $e \in E$ and $e' \in E$ there exists $t_1 \cup t_2$ soft open set $(F, E)$ and a $t_3 \cup t_4$ soft open set $(G, E)$ such that $e \in (F, E)$ and $e' \in (G, E)$ and $e' \in E$ and $e \in (G, E)$. Soft quad topological space $(X, t_1, t_2, t_3, t_4, t_5, t_6)$ is said to be pair wise soft $T_2$ space if $t_1 \cup t_2$ is soft $T_2$ space with respect to $t_1 \cup t_2$ and $t_2 \cup t_3$ is soft $T_2$ space with respect to $t_2 \cup t_3$.

**Definition 7.2:** In a soft quad topological space $(X, t_1, t_2, t_3, t_4, t_5, t_6)$
Proposition 17. If \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) be a soft quad topological space over \(X\) and \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) are soft \(T_4\) space then \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_4\) space.

Proof: Suppose \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is soft \(T_4\) space with respect to \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\). So according to definition for \(e_1, e_2, e_3 \in X_\nu\), \(e_1 \neq e_2\) there happens a \(t_1 \cup t_2\) soft open set \((F, E)\) and a \(t_1 \cup t_3\) soft open set \((G, E)\) such that \(e_1 \in (F, E)\) and \(e_2 \notin (F, E)\) or \(e_2 \in (G, E)\) and \(e_1 \notin (G, E)\) each \(t_1 \cup t_2\) soft closed set \((F, E)\) and a \(t_1 \cup t_3\) soft closed set \((G, E)\) such that \((F, E) \neq (G, E)\). There occurs \((F, E)\) and \((G, E)\) such that \((F, E) \neq (G, E)\). Hence \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is soft \(T_4\) space.

Proposition 18. Let \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) be a soft quad topological space over \(X\) and \(Y\) be a non-empty subset of \(X\) if \((Y, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_3\) space.

Proof: First we prove that \((Y, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_3\) space. Let \(e_1, e_2, e_3 \in X_\nu\), \(e_1 \neq e_2\) if \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft space then this implies that \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_3\) space. So there exists \(t_1 \cup t_2\) soft open set \((G, E)\) such that \(e_1 \in (F, E)\) and \(e_2 \notin (F, E)\) or \(e_2 \in (G, E)\) and \(e_1 \notin (G, E)\) now \(e_2 \in Y\) and \(e_1 \notin (G, E)\). Hence \(e_2 \in Y\) and \(\nu\) for \((F, E)\) then \(e_2 \notin \nu\) for \((G, E)\) for some \(a \in \nu\). This means that \(a \in E\) then \(e_2 \notin \nu \cap \nu\) for some \(a \in E\).

Now, we prove that \((Y, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_3\) space. Now, we prove that \((Y, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is pair wise soft \(T_3\) space. Let \(e_1 \in Y\) and \(e_2 \in (Y, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) be soft closed set in \(Y\) such that \(e_1 \in (G, E)\) where \(G, E\) is soft \(\tau_1 \cup \tau_2\) open set \((G, E)\) and \(G, E\) is soft \(\tau_1 \cup \tau_3\) open set \((G, E)\). Hence \((X, t_1, t_2, t_3, \tau, e, F, Y, \nu)\) is soft \(T_3\) space.

7. Conclusion

A soft set with single specific topological structure is unable to accept the duty to construct the whole theory. So to make the theory rich, some superfluous structures on soft set has to be announced. It makes, it more springy to develop the soft topological spaces with its countless applications. In this respect we introduce strong topological structure known as soft quad topological structure in this article.

